

UNIVERSITÉ DE MONTRÉAL

ON BINARY
MODE CHOICE/ASSIGNMENT MODELS

by
Michael Florian and Heinz Spiess

CENTRE DE RECHERCHE SUR LES TRANSPORTS



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This research was supported by a grant from the Natural Sciences
and Engineering Council Canada.

Université de Montréal
Centre de recherche sur les transports - Publication #177
Département d'informatique et de
recherche opérationnelle - Publication #374

Juillet 1980; révisé en février 1982

Dépôt légal - 2e trimestre 1982

RÉSUMÉ

Nous considérons le modèle d'équilibre de trafic à deux modes, le transport privé et le transport en commun, qui incorpore un modèle de choix modal agrégé par zone. Nous reformulons ce modèle spécial d'équilibre de réseau dans la forme d'une inéquation variationnelle et nous démontrons que les demandes et les coûts de transport entre les origines et les destinations, les flots et les coûts des liens sont uniques, quand certaines conditions suffisantes sont satisfaites. Nous démontrons, en suite l'existence de certains problèmes équivalents d'optimisation de certaines versions particulières de ce problème, et nous étudions la vérification de conditions suffisantes pour la convergence de méthodes de diagonalisation qui sont utilisées pour obtenir la solution de ce modèle.

ABSTRACT

We consider the two mode equilibrium road and transit assignment model which incorporates a zonal aggregate mode choice model. We reformulate this special structure network equilibrium model as a variational inequality and demonstrate that the origin to destination demands and travel costs, link flows and link travel costs are unique when appropriate sufficient conditions are satisfied. Then we demonstrate the existence of equivalent optimization formulations of special versions of this problem and study sufficient conditions for the convergence of diagonalization methods that are used to obtain solutions for this model.

The purpose of this work is to present in a unified way the work of Florian [9], Abdulaal and LeBlanc [3] and Fisk and Nguyen [8] on the two mode equilibrium road and transit assignment model, by casting the model in the form of a variational inequality. This formulation, which is inspired by the work of Smith [17] on the single mode asymmetric cost network equilibrium model and of Dafermos [6] on the multi-mode variable demand network equilibrium problem, permits the demonstration of the uniqueness of origin to destination demands and travel costs, link flows and link travel times under relatively weak sufficient conditions. We consider as well the special instance of this model when the costs of both modes depend only on the auto flows and, by deriving a variational inequality adapted for this cost structure, we indicate when the solution uniqueness may be expected. Then we use the variational inequality formulation to obtain, under various assumptions, equivalent convex cost optimization problems and to indicate sufficient conditions for the convergence of the algorithms that were used to solve this problem ([9], [3]), which may be characterized as diagonalization or relaxation methods.

Aashtiani and Magnanti [2] (see also [1]) formulated the general multi-mode variable demand network equilibrium problem as a nonlinear complementarity problem and demonstrated several existence theorems. We refer to some of their results in this paper.

The paper is organized in the following way. First we state the model and reformulate it as a variational inequality. Then we consider the conditions that ensure uniqueness of demands, origin to

destination travel costs, link flows and link travel times. Next we derive the equivalent convex cost optimization problems under the appropriate symmetry assumptions on the link user cost functions and study the convergence of diagonalization algorithms to solve the general version of this model. We consider as well the special instance of this model when the link user cost functions are 'nested', that is, the auto travel costs do not depend on the transit flows, but the transit travel costs depend on the flows of both modes.

1. THE MODEL

The network considered permits the flow of vehicles and transit passengers on links. The transit vehicles follow fixed itineraries. The nodes n , $n \in N$, represent origins, destinations and intersections of links; the links a , $a \in A$, represent the road and transit infrastructure of the urban area. Each link may be accessible to private and transit vehicles. Transit passengers may use only those links that carry transit vehicles. The modes are designated by index m which is 1 for the auto mode and 2 for the transit mode. The individual user costs s_a^m for travel by mode m on arc a are given by user cost functions that may depend on the flow of both modes, v_a^1 and v_a^2 , but are separable by link as follows

$$s_a^m = s_a^m(v_a), m = 1, 2 \text{ where } v_a = (v_a^1, v_a^2), a \in A \quad (1)$$

The user cost functions $s_a(v_a)$, where $s_a(v_a)$ is the two vector of user cost functions, are assumed to be monotone

$$(s_a(v'_a) - s_a(v''_a))^T(v'_a - v''_a) \geq 0, a \in A \quad (2)$$

continuous and differentiable.

The origin to destination demands, g_i^m , $i \in I$, where I is the set of origin/destination (O/D) pairs, for each mode m may use directed paths k , $k \in K_i^m$, where K_i^m is the set of paths, ($K_i^m \neq \emptyset$), available for mode m and O/D pair i . The total origin to destination demands by both modes are given by a constant matrix $\bar{g}$, where

$$g_i^1 + g_i^2 = \bar{g}_i, \quad i \in I \quad (3)$$

The flows on paths k , h_k , satisfy conservation of flow and nonnegativity.

$$\left. \begin{array}{l} \sum_{k \in K_i^m} h_k = g_i^m, \quad i \in I, m=1,2 \\ \text{and} \quad h_k \geq 0, \quad \forall k \end{array} \right\} \quad (4)$$

The link flows v_a^m are given by

$$v_a^m = \sum_{i \in I} \sum_{k \in K_i^m} \delta_{ak} \cdot h_k, \quad m=1,2, \quad a \in A \quad (5)$$

$$\text{where } \delta_{ak} = \begin{cases} 1 & \text{if link } a \text{ belongs to path } k \\ 0 & \text{otherwise} \end{cases}$$

The cost of each path $s_k(v)$ is the sum of the user costs of the links in the path

$$s_k(v) = \sum_a \delta_{ak} \cdot s_a^m(v_a), \quad k \in K_i^m, \quad i \in I, \quad m=1,2 \quad (6)$$

Let $u_i^m(v)$ be the cost of the least cost path for all O/D pairs i and modes m

$$u_i^m(v) = \min_{k \in K_i^m} s_k(v), \quad i \in I, \quad m=1,2 \quad (7)$$

A mode choice function $G_i(w_i)$, that depends on the travel costs by the two modes via their difference $w_i, w_i = u_i^1(v) - u_i^2(v)$, gives the proportion of trips that occur by mode 1, that is $g_i^1 = \bar{g}_i \cdot G_i(w_i)$. It is assumed that $G_i(w_i)$ is a strictly decreasing function with inverse $W_i(g_i^1/\bar{g}_i)$. Since $\bar{g}_i$ is constant for a given i we refer to the inverse function as $W_i(g_i^1)$.

The binary mode choice/assignment model is formulated by supposing that no traveller has the incentive to change mode

$$u_i^{1*} - u_i^{2*} = W_i(g_i^{1*}), \quad i \in I \quad (8)$$

and that for each mode the path choice satisfies Wardrop's [18] user optimized behavioral principle.

$$s_k^* - u_i^{m*} \begin{cases} = 0 & \text{if } h_k^* > 0 \\ \geq 0 & \text{if } h_k^* = 0 \end{cases}, \quad k \in K_i^m, \quad i \in I, \quad m=1,2 \quad (9)$$

subject to the feasibility constraints (3), (4) and (5). The notation $*$ indicates equilibrium values of the flows, demands and O/D costs.

Before casting (8) and (9) in the form of a variational inequality, we note the following

Lemma 1. (9) are equivalent to

$$s_{k_1}^* \leq s_{k_2}^*, \quad \text{if } h_{k_1}^* > 0, \quad k_1, k_2 \in K_i^m, \quad i \in I, \quad m=1,2 \quad (10)$$

Proof. Since $s_{k_1}^* = u_i^{m*}$ if $h_{k_1}^* > 0$ but $s_{k_2}^* \geq u_i^{m*}$ if $h_{k_2}^* \geq 0$, (10) follows directly from (9). Now let (10) be satisfied. It follows then that for all k_1 such that $h_{k_1}^* > 0$, $u_i^{m*} = s_{k_1}^*$ and thus for all k_2 such that $h_{k_2}^* \geq 0$, $u_i^{m*} \leq s_{k_2}^*$, thus (9) holds. $\square$

We use this observation in proving the following result.

Theorem 1. The conditions (8) and (9) are equivalent to

$$s(v^*)^T(v-v^*) - W(g^{1*})^T(g^1 - g^{1*}) \geq 0, \quad v(v, g^1) \text{ feasible} \quad (11)$$

where $s(v)$ is the vector of user cost functions and $W(g^1)$ is the vector of inverse mode choice functions, over the feasible set defined by (3), (4) and (5).

Proof. First we show that (8) and (9) imply (11). For a given O/D pair i and path k for each mode it follows from (9) that

$$(s_k^* - u_i^{m*})(h_k - h_k^*) \geq 0, \quad k \in K_i^m, \quad m=1,2, \quad i \in I$$

By summing over $k \in K_i^m$ and $m=1,2$, one obtains

$$\sum_{k \in K_i^1} s_k^* (h_k - h_k^*) + \sum_{k \in K_i^2} s_k^* (h_k - h_k^*) \geq u_i^{1*} (g_i^1 - g_i^{1*}) + u_i^{2*} (g_i^2 - g_i^{2*}), \quad i \in I.$$

Now $g_i^2 = \bar{g}_i - g_i^1$ and $g_i^{2*} = \bar{g}_i - g_i^{1*}$, therefore, after making this substitution and summing over $i \in I$, it follows that

$$\sum_{i \in I} \sum_{k \in K_i^1} s_k^* (h_k - h_k^*) + \sum_{i \in I} \sum_{k \in K_i^2} s_k^* (h_k - h_k^*) \geq \sum_{i \in I} (u_i^{1*} - u_i^{2*}) (g_i^1 - g_i^{1*})$$

By changing the order of summation on the left hand side, using (6) and applying (8) one finally obtains

$$\sum_a s_a^1 (v_a^1 - v_a^{1*}) + \sum_a s_a^2 (v_a^2 - v_a^{2*}) - \sum_i W_i(g_i^{1*}) (g_i^1 - g_i^{1*}) \geq 0$$

Thus, in vector notation, we find that the above inequality is identical to (11).

We show next that (11) implies (8) and (9). Let (v^*, g^{1*}) be any solution of (11). First we demonstrate that (11) implies (9). Let $k_1 \in K_i^m$ be any used path for mode m , $h_k^* > 0$, and let $k_2 \in K_i^m$ be any alternative path for the same mode. We define now (v, g^1) to be the same as (v^*, g^{1*}) except for the two paths k_1 and k_2 , for which $h_{k_1} = h_{k_1}^* - \delta$, $h_{k_2} = h_{k_2}^* + \delta$, where $0 < \delta \leq h_{k_1}^*$. We note that $g^1 = g^{1*}$ since k_1 and k_2 are paths for the same mode. Applying (11) yields $s_{k_1}^* \cdot (-\delta) + s_{k_2}^* \cdot \delta \geq 0$ and division by δ results in (10). Since (10) implies (9), by Lemma 1, it follows that (11) implies (9). We conclude the proof by showing that (11) and (9) imply (8). Since we have assumed that $G_i(w_i)$ is invertible, this function is strictly decreasing and hence $g_i^{m*} > 0$, $m=1,2$. Then there exists at least one path for each mode, $k_1 \in K_i^1$, $k_2 \in K_i^2$, such that $h_{k_1}^* > 0$, $h_{k_2}^* > 0$. As before we construct (v, g^1) that differs from (v^*, g^{1*}) only in the flows of the two paths k_1 and k_2 , for which $h_{k_1} = h_{k_1}^* - \delta$, $h_{k_2} = h_{k_2}^* + \delta$, where $0 < |\delta| \leq \min(h_{k_1}^*, h_{k_2}^*)$. Note that $g_i^1 = g_i^{1*} - \delta$, $g_i^2 = g_i^{2*} + \delta$. Applying (11) yields $s_{k_1}^* \cdot (-\delta) + s_{k_2}^* \cdot \delta - W(g_i^{1*})\delta \geq 0$, which, after division by δ and use of (9) results in $-u_i^{1*} + u_i^{2*} - W(g_i^{1*}) \geq 0$ for $\delta > 0$, and $-u_i^{1*} + u_i^{2*} - W(g_i^{1*}) \leq 0$ for $\delta < 0$, hence (8) is satisfied. $\square$

Since this model is a special case of the general model studied in [2], existence of a solution to (11) follows from the results obtained in that work.

In the following section we consider conditions which ensure uniqueness of the mode demands, link flows and O/D travel costs.

2. UNIQUENESS

A mode choice function $G_i(w_i)$, which satisfies rational assumptions about the behavior of travellers in choosing their mode, is a strictly decreasing function of w_i . We assumed earlier that this property holds, that is

$$\text{for } w' \neq w'', (G(w') - G(w''))^T (w' - w'') < 0 \quad (12)$$

This implies that $-G(w)$ is a strictly monotone mapping and, therefore

$$\text{for } g^{1'} \neq g^{1''}, (W(g^{1'}) - W(g^{1''}))^T (g^{1'} - g^{1''}) < 0 \quad (13)$$

or that $-W(g^1)$ is a strictly monotone mapping.

We recall that the user cost functions were assumed to be monotone, as stated in (2). A most satisfying property, for planning applications, is that when the link user cost functions are monotone and the mode choice functions are strictly decreasing functions of the difference in travel costs, the demands, link travel costs and O/D travel costs are unique.

Theorem 2. (12) and (13) imply that g_i^{1*} , w_i^* , $i \in I$ are unique, as well as $s_a^m(v^*)$, $a \in A$, $m = 1, 2$ and u_i^{m*} , $i \in I$, $m = 1, 2$.

Proof. Assume that there exist two distinct solutions $(v', g^{1'})$ and $(v'', g^{1''})$, which both satisfy (11). By writing (11), once with $v = v'$, $g^1 = g^{1'}$ and $v^* = v''$, $g^{1*} = g^{1''}$ and once with $v = v''$, $g^1 = g^{1''}$ and $v^* = v'$, $g^{1*} = g^{1'}$, one obtains, by adding the two inequalities

$$(s(v') - s(v''))^T (v' - v'') - (W(g^{1'}) - W(g^{1''}))^T (g^{1'} - g^{1''}) = 0 \quad (14)$$

However, by (2) and (13) it follows that (14) is satisfied if and only if each term is equal to zero. Thus

$$(s_a(v'_a) - s_a(v''_a))^T (v'_a - v''_a) = 0, \quad a \in A \quad (15)$$

and

$$(W_i(g_i^{1'}) - W_i(g_i^{1''}))^T (g_i^{1'} - g_i^{1''}) = 0, \quad i \in I \quad (16)$$

From (16) it follows that $g_i^{1'} = g_i^{1''}$, $i \in I$ and hence

g_i^{1*} , w_i^* , $i \in I$ are unique. From (15) it follows that $s_a(v'_a) = s_a(v''_a)$, $a \in A$, hence the link travel costs are unique as well. Since u_i^{1*} , u_i^{2*} , $i \in I$, are the lengths of shortest paths on the respective networks, it follows that they are unique as well. $\square$

We note that the link flows need not be unique. If the user cost functions are strictly monotone (see also [2],[5],[16]), that is, for $v' \neq v''$, $(s(v') - s(v''))^T (v' - v'') > 0$ (17)

then the link flows are unique.

Corollary 1. If (17) holds as well, then v^* is unique as well.

Proof. From (15) it follows that $v'_a = v''_a$ if (17) holds, since $s(v)$ is separable by link. $\square$

We turn our attention to the verification of the monotonicity conditions (2) and (12). First, we would like to point out that the verification of (12) is trivial.

In the analysis of the general model, where the demand function may

depend on travel costs between all O/D pairs, the verification of the equivalent condition for uniqueness is more difficult ([2], [7]). As an example, the conditions of Theorem 2 are satisfied for logit mode choice functions such as

$$G_i(w_i) = 1 / (1 + \exp(\alpha + \beta w_i)) , \text{ for } \beta > 0 \quad (18)$$

or

$$G_i(w_i) = 1 / (1 + \exp(\alpha + \beta u_i^1 - \gamma u_i^2)) , \text{ for } \beta, \gamma > 0 \quad (19)$$

since the user costs on the links may be appropriately scaled to convert (19) into a similar form to (18).

The verification of (2) is relatively easy since the Jacobian matrix $B, B = \frac{\partial s(v)}{\partial v}$, has a block diagonal structure for the two mode problem, due to the separability of the user cost functions by link, where each block corresponds to the link a and is

$$B_a = \frac{\partial s_a(v_a)}{\partial v_a} = \begin{bmatrix} b_a^{11} & b_a^{12} \\ b_a^{21} & b_a^{22} \end{bmatrix} , \quad a \in A \quad (20)$$

where $b_a^{11} = \frac{\partial s_a^1(v_a)}{\partial v_a^1}$, for instance.

Theorem 3. The user cost mapping $s(v)$ is monotone if and only if, for all v ,

$$4b_a^{11} b_a^{22} \geq (b_a^{12} + b_a^{21})^2 , \quad a \in A \quad (21)$$

Proof. It is well known (see [15]) that $s(v)$ is monotone on the set of feasible flows if and only if $\frac{\partial s(v)}{\partial v}$ is positive semi-definite for all feasible v . Since B_a

in not symmetric in general, it follows that each block B_a must satisfy the condition that $B_a + B_a^T$ is a positive semi-definite matrix. Thus if (21) is satisfied then $s(v)$ is monotone $\square$.

A direct consequence of the above is the following.

Corollary 2. The user cost mapping $s(v)$ is strictly monotone on the set of feasible flows if and only if, for all v

$$4b_a^{11} b_a^{22} > (b_a^{12} + b_a^{21})^2, \quad a \in A \quad (22)$$

We consider now the special instance of this model when the user cost functions are nested, that is, the user costs for mode 1 do not depend on mode 2, however the user costs for mode 2 depend on both mode 1 and mode 2.

$$\left. \begin{aligned} s_a^1 &= s_a^1(v_a^1) \\ s_a^2 &= s_a^2(v_a^1, v_a^2) \end{aligned} \right\} \quad a \in A \quad (23)$$

In this case $b_a^{12} = 0$ and it follows that

Corollary 3. The user cost mapping $s(v)$ defined by (23) for all links a is monotone (strictly monotone) on the set of feasible flows if and only if, for all v

$$4b_a^{11} b_a^{22} \geq (b_a^{21})^2 \quad (4b_a^{11} b_a^{22} > (b_a^{21})^2), \quad a \in A.$$

The structure of the user cost functions suggested in [9] and calibrated for a particular application in [10] are nested however the user costs for mode 2 depend only on mode 1, that is

$$\left. \begin{aligned} s_a^1 &= s_a^1(v_a^1) \\ s_a^2 &= s_a^2(v_a^1) \end{aligned} \right\} \quad a \in A, \quad \text{where } s_a^1(v_a^1), s_a^2(v_a^1) \text{ are strictly increasing functions} \quad (24)$$

In this case $b_a^{12} = b_a^{22} = 0$, and it follows that

Corollary 4. The user cost mapping $s(v)$ defined by (24) for all links a is not monotone.

Proof. It is trivial to verify that $B_a^T + B_a$ has a negative determinant, hence it is not positive semi-definite $\square$

The result of Theorem 1 does not apply then for user cost functions (24) and multiple equilibria may exist as demonstrated by the following example. The network consists of one O/D pair connected by one link which is accessible by both modes. $g^1 = v^1 = x$; $g^2 = \bar{g} - x$; $s^1(x) = x$; $s^2(x) = x^2$. The mode choice function is $G(w) = -w + 3/4$, $-17/4 \leq w \leq 3/4$, and $w(x) = -x + 3/4$, $0 \leq x \leq 5$. Each x that satisfies $s^1(x) - s^2(x) = w(x)$ is an equilibrium solution for this model and this condition is satisfied for both $x = .5$ and $x = 1.5$.

In order to analyse the uniqueness properties of the mode choice/assignment model with user cost functions defined by (24) we derive another way of stating the variational inequality (11), by considering it in the form

$$\sum_{i \in I} \sum_{k \in K_i^1} s_k^*(h_k - h_k^*) + \sum_{i \in I} \sum_{k \in K_i^2} s_k^*(h_k - h_k^*) - \sum_{i \in I} w_i(g_i^{1*})(g_i^1 - g_i^{1*}) \geq 0 \quad (25)$$

over the feasible set (3), (4), (5). For this purpose we define

$$\left. \begin{aligned} u_i^1(g^1) &= \min_{k \in K_i^1} (s_k(v^1)) \\ u_i^2(g^1) &= \min_{k \in K_i^2} (s_k(v^1)) \end{aligned} \right\} \quad i \in I \quad (26)$$

where v^1 is the unique solution of the single mode network equilibrium problem

$$\text{Min } Z = \sum_a \int_0^{v_a^1} s_a^1(x) dx$$

$$\text{Subject to } \sum_{k \in K_i^1} h_k = g_i^1, \quad i \in I \quad (27)$$

$$h_k \geq 0, \quad k \in K_i^1, \quad i \in I$$

Then (25) may be restated as

$$\sum_{i \in I} u_i^1(g_i^{1*})(g_i^1 - g_i^{1*}) - \sum_{i \in I} u_i^2(g_i^{1*})(g_i^1 - g_i^{1*}) - \sum_{i \in I} w_i(g_i^{1*})(g_i^1 - g_i^{1*}) \geq 0 \quad (28)$$

by using the fact that $g_i^1 + g_i^2 = \bar{g}_i$, $i \in I$, as done in the proof of Theorem 1. Thus we have proved

Theorem 4. When the user cost functions are given by (24), the variational inequality (11) is equivalent to (28) over the feasible set defined by $0 \leq g_i \leq \bar{g}_i$, $i \in I$.

(28) may be restated as

$$\sum_{i \in I} F_i(g_i^{1*})(g_i^1 - g_i^{1*}) \geq 0 \quad (29)$$

where $F_i(g_i^1) = u_i^1(g_i^1) - u_i^2(g_i^1) - w_i(g_i^1)$. It is important to establish the following.

Theorem 5. The functions $F_i(g_i^1)$ are continuous functions of g_i^1 if $w_i(g_i^1)$ are continuous functions of g_i^1 , $i \in I$.

Proof. Hall [12] demonstrated that (v^1, u^1) vary continuously with g^1 for the problem (27). Since $s_a^2(v_a^1)$ are continuous functions of v_a^1 and $u_i^2(g^1) = \min_{k \in K_i} s_k(v^1)$, it follows that $u^2(g^1)$ varies continuously with g^1 as well. Thus, if $W_i(g_i^1)$ is a continuous function of g_i^1 , $F_i(g^1)$ is a continuous function of g^1 . $\square$

We note that although $F_i(g^1)$ is continuous in g^1 it is not continuously differentiable and that the proof of Theorem 5 is along the same lines as the proof of Lemma 1 in [8]. We state the following without proof.

Theorem 6. g^{1*} is unique for (29) if $F(g^1)$ is a strictly monotone mapping of g^1 .

An immediate consequence is

Corollary 5. If g^{1*} is unique, then v^{1*} is unique as well if $s_a^1(v_a^1)$, $a \in A$, are strictly increasing functions.

Proof. Since v^{1*} is the solution of (27) with $g_i^1 = g_i^{1*}$, $i \in I$, it is unique since Z is strictly convex if $s_a^1(v_a^1)$, $a \in A$ are strictly increasing functions. $\square$

It is not easy to establish that $F(g^1)$ is strictly monotone in g^1 , however (29) provides a framework for interpreting the work of Fisk and Nguyen [8], who analysed the uniqueness of the solution of the special instance of this mode choice/assignment model which incorporates a zonal aggregate logit mode choice function. They

established that $u^1(g^1)$ is a monotone mapping (see Theorem 2 in [8]) and then showed, using empirical arguments that for the particular implementation of the model reported in [10], the mapping $(-u^2(g^1) - W(g^1))$ is expected to be monotone. Although their condition is not stated explicitly in terms of the inverse mode choice mapping $W(g^1)$, the correspondence with their analysis (see Lemma 2 in [8]), is immediate.

In summary the monotonicity of $F_i(g^1)$, $i \in I$, may be established by demonstrating one of the following:

- (i) $(u^1(g^1) - u^2(g^1) - W(g^1))$ strictly monotone (30)
- (ii) $(-u^2(g^1) - W(g^1))$ strictly monotone, since $u^1(g^1)$ is monotone
- (iii) $(u^1(g^1) - u^2(g^1))$ monotone since $-W(g^1)$ is strictly monotone
- (iv) $u^2(g^1)$ monotone since $u^1 - W(g^1)$ is strictly monotone.

Conditions (ii), (iii) and (iv) are stronger than (i), and therefore imply it. (iii) and (iv) are probably the most difficult to demonstrate. Fisk and Nguyen [8] chose to work with (ii) and, in general, the demonstration of (i) is not evident either. Roughly speaking, the monotonicity of $F_1(g^1)$ is likely to be satisfied in practice, since the inverse mode choice functions are quite steep, that is, a change in g^1 induces a much smaller change in $\Delta_i(g^1) = u_i^1(g^1) - u_i^2(g^1)$, than in the corresponding $W_i(g^1)$. This is illustrated by means of the following simple examples.

The network consists of one O/D pair and two links. The user cost functions are $s_1^1(v_1) = 5 + 2v_1^1$; $s_2^1(v_2) = 1 + 4v_2^1$;

$s_1^2(v_1) = s_1^1(v_1) + (s_1^1(v_1))^{1/2}$. The mode choice function is

$g^1 = 6/(1+\exp(.5+.2(u^1 - u^2)))$ and mode 2 may only use link 1.

The network is shown in Figure 1. For this example $u^2(g^1)$ and $\Delta(g^1)$ are not monotone, however $(-u^2(g^1) - W(g^1))$ as well as $F(g^1)$ are strictly monotone as shown in Figure 2. Therefore (ii) applies and it follows that $F(g^1)$ is strictly monotone. If the mode choice function is changed to $g^1 = 6/(1+\exp(u^1 - u^2))$ then $(-u^2(g^1) - W(g^1))$ is not monotone, however $F(g^1)$ is strictly monotone nevertheless, as shown in Figure 3. Thus, for both examples the solution is unique, even though the mode choice function in the second example is somewhat unrealistic.

3. EQUIVALENT OPTIMIZATION PROBLEMS AND SOLUTION ALGORITHMS

We turn our attention now to the existence of equivalent optimization problems for the mode choice/assignment model. (11) permits the direct derivation of results that are due to Abdulaal and LeBlanc [3]. As before let B denote the Jacobian of $s(v)$.

Theorem 7. If B is symmetric, the solution of (11) may be obtained by solving

$$\text{Min } Z = \int_0^v s(x)dx - \sum_{i \in I} \int_0^{g_i^1} w_i(y)dy \quad (30)$$

subject to (3), (4) and (5).

Proof. If B is symmetric, then $s(v)$ is integrable, and (11) may be interpreted as the optimality condition for the above problem, that is, that the directional derivative of Z is larger than or

equal to zero. (It suffices to recall that $f(x)$ convex, $x \in \Omega$, Ω a convex set, is minimized at x^* if and only if $\nabla f(x^*)(y-x^*) \geq 0$, $y \in \Omega$). $\square$

We note that if $s(v)$ is strictly monotone as well, then (30) is a strictly convex function. The following consequence of Theorem 7, (see [3]) is useful for certain algorithms that were suggested for computing a solution to the mode choice/assignment problem.

Corollary 6. If $s_a^1(v_a) = s_a^1(v_a^1)$ and $s_a^2(v_a) = s_a^2(v_a^2)$ the solution of (11) is equivalent to

$$\text{Min } Z = \sum_a \int_0^{v_a^1} s_a^1(x) dx + \sum_a \int_0^{v_a^2} s_a^2(x) dx - \sum_{i \in I} \int_0^{g_i^1} w_i(y) dy \quad (31)$$

Subject to (3), (4) and (5).

The reformulation of the mode choice/assignment problem as a variational inequality permits the application of any iterative method that has proven convergence properties. Thus, it is possible, for instance, to adapt the projection method proposed by Dafermos in [6] for solving this problem. The first algorithms that were used to solve the mode choice/assignment problem by Florian [9] and Abdulaal and LeBlanc [3] are based on an approach which is intuitively appealing and, in retrospect, is akin to the Jacobi type iteration (see [15,16]) for solving linear or nonlinear equations. This method proceeds as follows: given a feasible solution $(\bar{v}, \bar{g}^1)$, a new problem is defined by fixing the effect of the flows v^1 on the costs $s^2(v)$ and vice versa as follows:

$$\left. \begin{aligned} s_a^1(v_a) &= s_a^1(v_a^1, \tilde{v}_a^2) \\ s_a^2(v_a) &= s_a^2(\tilde{v}_a^1, v_a^2) \end{aligned} \right\}, \quad a \in A \quad (32)$$

This renders $s(v, \tilde{v})$ a diagonal mapping in v (Recall, [15], that a nonlinear mapping $F(x)$, $x \in R^n$, is diagonal if, for $i = 1, \dots, n$ its i_{th} component $f_i(x)$ is only a function of x_i) and by Corollary 6 there exists an equivalent convex cost minimization problem for (11) in this case. Thus, we shall refer to this method as the diagonalization method. In the numerical analysis literature such methods are also denoted as relaxation methods.

A sequence of convex cost minimization problems are solved by standard methods [13], [14], until a satisfactory solution is obtained. This iterative method may be stated as:

Step 0 (Initialization). Find feasible $(\tilde{v}, \tilde{g}^1)$

Step 1 (Diagonalize user cost functions). Apply (32).

Step 2 (Solve diagonalized problems). Find $(\hat{v}, \hat{g}^1)$ that

$$\text{Min } Z = \sum_a \int_0^{v_a^1} s_a^1(x, \tilde{v}_a^2) dx + \sum_a \int_0^{v_a^2} s_a^2(\tilde{v}_a^1, x) dx - \sum_{i \in I} \int_0^{g_i^1} w_i(y) dy$$

subject to (3), (4) and (5).

Step 3 (Convergence test). If $(\hat{v}, \hat{g}^1)$ is sufficiently close to $(\tilde{v}, \tilde{g}^1)$ stop; otherwise

Step 4. Set $(\tilde{v}, \tilde{g}^1) = (\hat{v}, \hat{g}^1)$ and return to step 1.

It is fairly easy to show (see for instance [3]), that

if the algorithm terminates, it terminates at a solution of the mode choice/assignment problem. As in practice the conditions that ensure uniqueness are usually satisfied, the solution obtained is the only one possible.

We turn our attention briefly to the nested cost model. The algorithm suggested in [9] is identical to that stated above, except that it was suggested for nested costs such as (24). Thus Step 2 specializes for that cost structure by solving the problem

$$\text{Min } Z = \sum_a \int_0^{v_a^1} s_a^1(x) dx - \sum_{i \in I} \int_0^{g_i^1} w_i(y + \bar{u}_i^2) dy$$

since the costs $s_a^2(\bar{v}_a^1)$ are constant at this step of the iterative method. Now $w_i(g_i^1) + \bar{u}_i^2$ is the inverse of the mode choice function $G_i(w_i)$ when $u_i^2 = \bar{u}_i^2$, since $G_i(u_i^1 - \bar{u}_i^2) = g_i^1$, thus $w_i(g_i^1) = u_i^1 - \bar{u}_i^2$ and $u_i^1 = w_i(g_i^1) + \bar{u}_i^2$. The relation between the algorithms proposed in [9] and [3] is evident.

Until recently, the convergence of this iterative method was not supported by rigorous arguments. In recent articles, Dafermos [7] provides a sufficient condition for the global convergence of this method and Florian and Spiess [11] indicate a sufficient condition for the local convergence of this method, which is that in a neighbourhood of v^* the condition

$$\|I - D^{-\frac{1}{2}} B D^{-\frac{1}{2}}\|_2 < 1 \quad (33)$$

be satisfied, where D is diagonal of the Jacobian B . Some insight

into the stringency of this condition, which is of course weaker than the condition for global convergence in [7], may be gained by the evaluation of the following conditions on B , that are relatively easy to verify.

Theorem 8. If

$$|b_a^{m,\bar{m}}| < (b_a^{11} \cdot b_a^{22})^{\frac{1}{2}}, \quad \begin{matrix} m = 1, 2 \\ \bar{m} = 3-m \end{matrix}, \quad a \in A \quad (34)$$

then (33) is satisfied.

Proof. Recall that for a matrix P the norm $\|P\|_2^2 = \rho(P^T P)$ where ρ denotes the largest eigenvalue of $(P^T P)$. Therefore

$$\|I - D_a^{-\frac{1}{2}} B_a D_a^{-\frac{1}{2}}\|_2 = \max_m (b_a^{m,\bar{m}})^2 (b_a^{11} \cdot b_a^{22}), \quad \forall a$$

If (34) is satisfied, this implies that

$$\|I - D^{-\frac{1}{2}} B D^{-\frac{1}{2}}\|_2 = \max_a \|I - D_a^{-\frac{1}{2}} B_a D_a^{-\frac{1}{2}}\|_2 < 1 \quad \square$$

We note that (34) is weaker than strict row and column dominance, which is

$$\left. \begin{array}{l} \text{(a)} \quad |b_a^{m,\bar{m}}| < |b_a^{m,m}| \\ \text{(b)} \quad |b_a^{\bar{m},m}| < |b_a^{\bar{m},\bar{m}}| \end{array} \right\} \quad \begin{matrix} m = 1, 2, \\ \bar{m} = 3 - m \end{matrix} \quad a \in A \quad (35)$$

Corollary 7. (35) implies (34).

Proof. By multiplying 35(a) by 35(b) for fixed a , one obtains (34). $\square$

For instance, the matrix $\begin{bmatrix} 4 & 3 \\ 5 & 8 \end{bmatrix}$ satisfies (34) but not (35).

We conclude this section by noting that (34) is not satisfied for user cost functions (24) since $b_a^{22} = 0$, $a \in A$. It seems appropriate to mention that global convergence conditions for this algorithm may be derived in another way, by using the approach of Fisk and Nguyen [8] who interpret it as a method of successive approximations for solving a system of nonlinear equations.

4. CONCLUSION

We have provided, by using the reformulation of the mode choice/assignment problem as a variational inequality, a unified way of presenting previous formulations and analyses ([9], [3] and [8]) of this model.

Although the convergence of the algorithm studied is ensured by conditions that are not always satisfied for all instances of this model, the application of this model is attractive when the conditions that ensure the uniqueness of the solution are satisfied. The applications of this method in practice, e.g. [16], have always resulted in satisfactory convergence.

ACKNOWLEDGEMENT

We would like to thank Caroline Fisk for several useful comments.

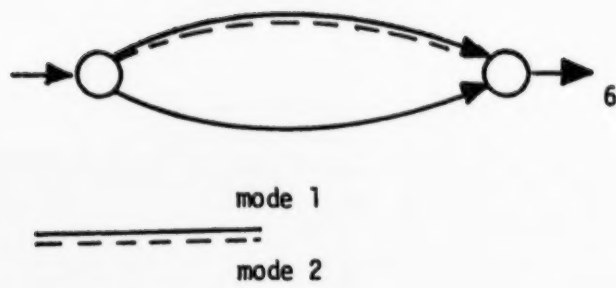


Figure 1

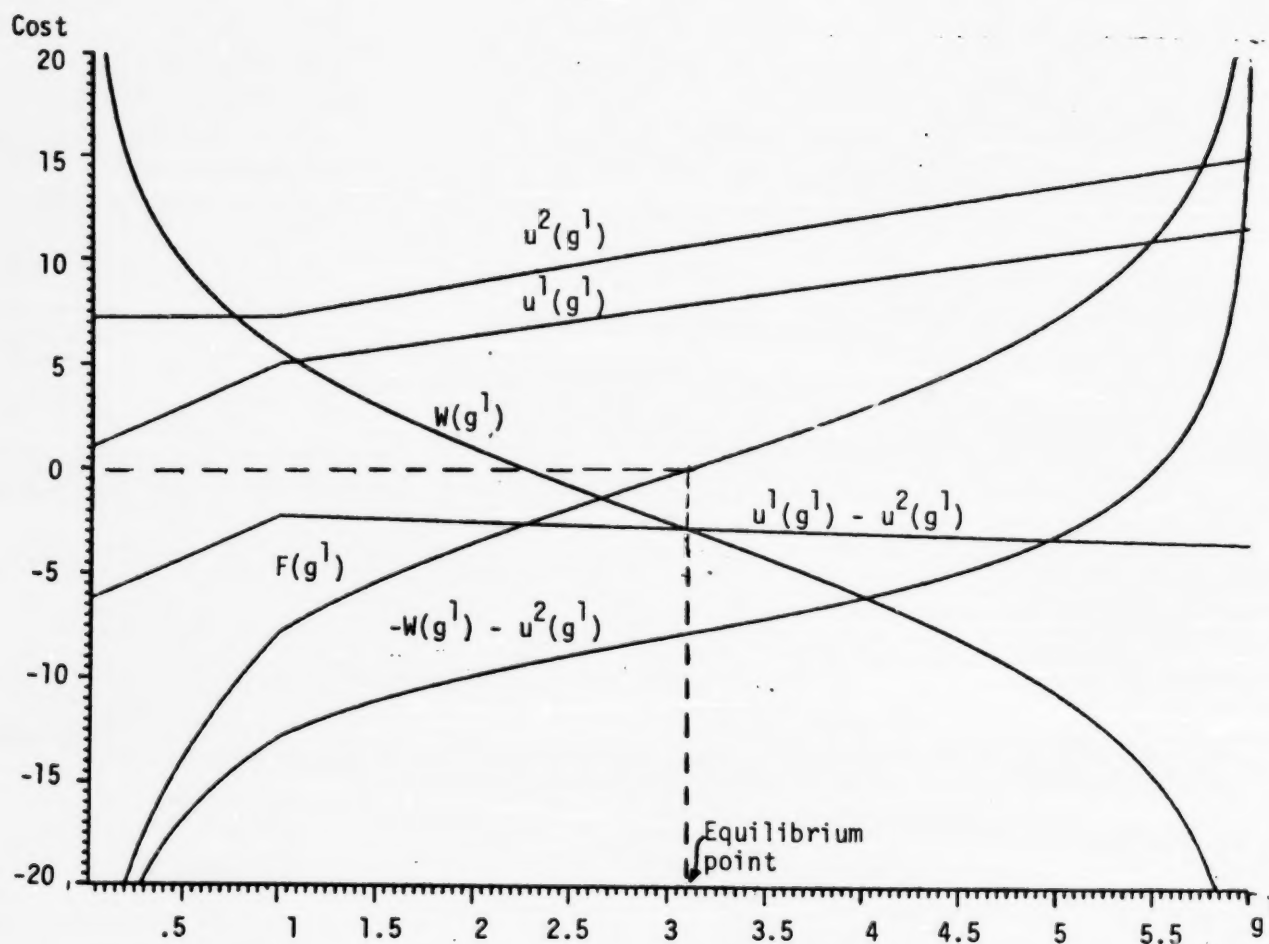


Figure 2

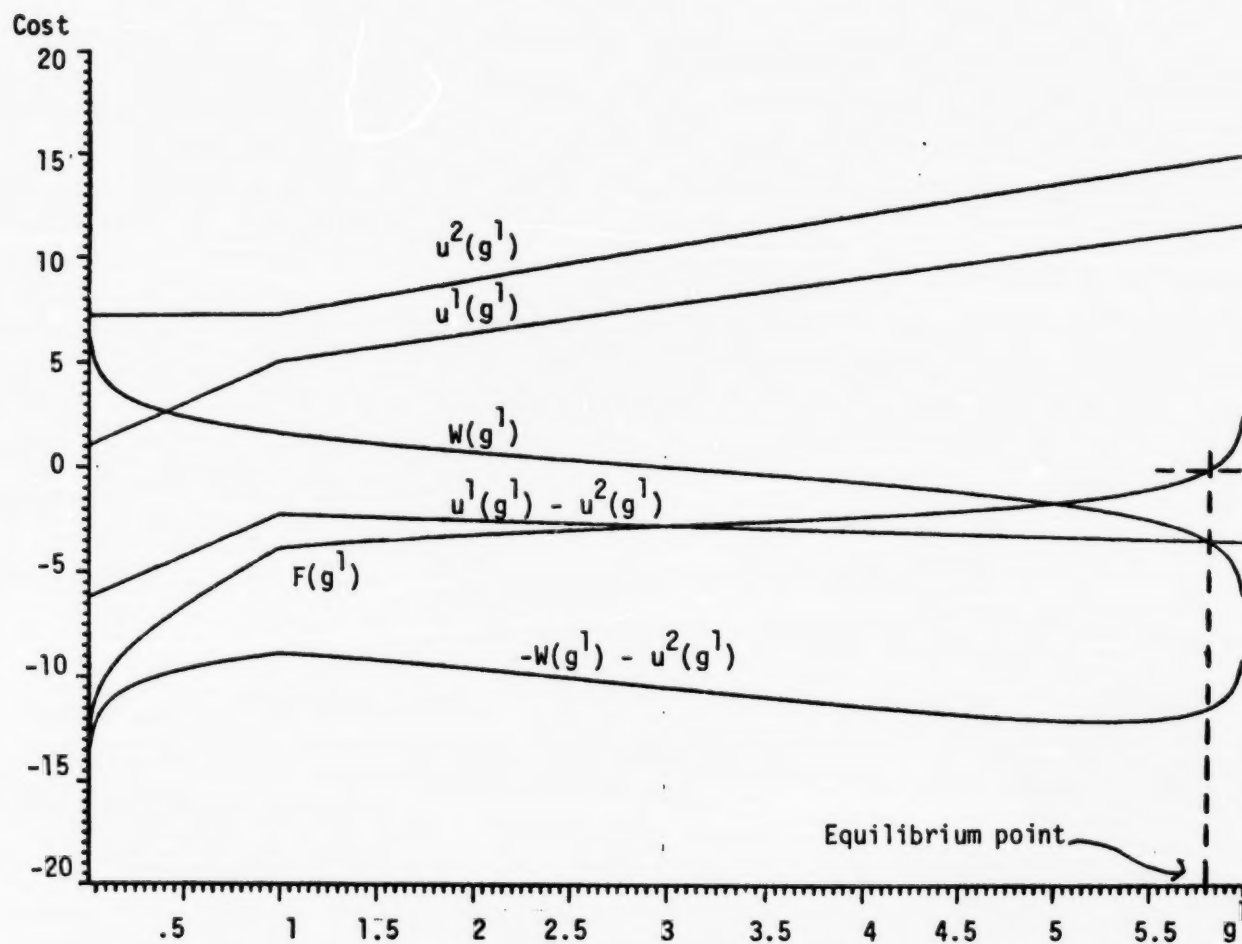


Figure 3

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